

# THE MATHEMATICS OF SPINPOSSIBLE

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## 1. INTRODUCTION

Spinpossible<sup>TM</sup> is played on  $3 \times 3$  board of tiles numbered from 1 to 9, each of which may be right-side-up or up-side-down. One possible starting position is the board:

5	7	6
2	1	9
7	8	3

The objective of the game is to return the board to the standard configuration:

1	2	3
4	5	6
7	8	9

This is accomplished by a sequence of *spins*, each of which rotates a rectangular region of the board by  $180^\circ$ . The goal is to minimize the number of spins used. The starting board above may be solved using two spins:

5	7	6
2	1	9
7	8	3

 $\implies$ 

1	2	6
4	5	9
7	8	3

 $\implies$ 

1	2	3
4	5	6
7	8	9

In this example, the first spin rotates the  $2 \times 2$  rectangle in the top left, and the second spin rotates the  $3 \times 1$  rectangle along the right edge. You can play the game online at <http://spinpossible.com>

In these notes we give a mathematical description of this game, and some of its generalizations, and consider various questions that naturally arise. Perhaps the most obvious is this: is it always possible to return a given board to the standard configuration with a sequence of spins? We shall see shortly that the answer is yes.

A more difficult question is the following: what is the maximum number of spins required to solve any board? An exhaustive search has found that 9 spins are always sufficient (and sometimes necessary), but no short proof of this fact is known.

## 2. A MATHEMATICAL DESCRIPTION OF THE GAME

We begin by defining the group  $\text{Spin}_{m \times n}$ , for a fixed pair of positive integers  $m$  and  $n$  with product  $N = mn$ . Let  $S_N$  denote the symmetric group on  $N$  letters with the action on the right (so the permutation  $\alpha\beta$  applies  $\alpha$  and then  $\beta$ ), and let  $V_N = (\mathbb{Z}/2\mathbb{Z})^N$  denote the additive group of  $N$ -bit vectors. For any vector  $\mathbf{v} = (v_1, \dots, v_N) \in V_N$  and permutation  $\alpha \in S_N$ , we use  $\mathbf{v}^\alpha = (v_{\alpha^{-1}(1)}, \dots, v_{\alpha^{-1}(N)})$  to denote the vector obtained by applying  $\alpha$  to  $\mathbf{v}$ .

**Definition 1.** The group  $\text{Spin}_{m \times n}$  is the set  $\{(\alpha, \mathbf{u}) : \alpha \in S_{mn}, \mathbf{u} \in V_{mn}\}$  under the operation  $(\alpha, \mathbf{u})(\beta, \mathbf{v}) = (\alpha\beta, \mathbf{u}^\beta + \mathbf{v})$ . Equivalently,  $\text{Spin}_{m \times n}$  is the wreath product  $\mathbb{Z}/2\mathbb{Z} \wr S_N$ .

Readers familiar with Coxeter groups will recognize  $\text{Spin}_{m \times n}$  as the hyperoctahedral group of degree  $N$  (the symmetry group of both the  $N$ -cube and the  $N$ -octahedron), equivalently, the Weyl group of type  $B_N$  (and  $C_N$ ). This group can also be represented using signed permutation matrices, but the representation as a wreath product is better suited to our purposes here. The definition of the group  $\text{Spin}_{m \times n}$  depends only on  $N = mn$ , however the integers  $m$  and  $n$  determine the set of generators we will be defining shortly, and these play a key role in the game (playing Spinpossible on a  $1 \times 9$  board would be much less interesting!).

A *board* is an  $m \times n$  array of uniquely identified elements called *tiles*, which we number from 1 to  $mn$ . Each tile may be oriented positively (right-side-up), or negatively (upside-down). The *positions* of the board are fixed locations, which for convenience we regard as unit squares in the plane, also numbered from 1 to  $mn$ , starting at the top left and proceeding left to right, top to bottom. The *standard board* has tile  $i$  in position  $i$ , with positive orientation.

There is a 1-to-1 correspondence between  $m \times n$  boards and elements of  $\text{Spin}_{m \times n}$ , but we will generally think of elements of  $\text{Spin}_{m \times n}$  as acting on the set of all  $m \times n$  boards as follows: the element  $(\alpha, \mathbf{v})$  first permutes the tiles by moving the tile in position  $i$  to position  $\alpha(i)$ , and then reverses the orientation of the tile in the  $i$ th position if and only if  $v_i = 1$ . Of course this is just the action of  $\text{Spin}_{m \times n}$  on itself.

The projection map  $\pi : \text{Spin}_{m \times n} \rightarrow S_N$  that sends  $(\alpha, \mathbf{u})$  to  $\alpha$  is a group homomorphism, and we have the short exact sequence

$$1 \longrightarrow V_N \longrightarrow \text{Spin}_{m \times n} \longrightarrow S_N \longrightarrow 1.$$

It is worth emphasizing that the projection from  $\text{Spin}_{m \times n}$  to  $V_N$  is *not* a group homomorphism (for  $N > 1$ ).

We now distinguish the elements of  $\text{Spin}_{m \times n}$  that correspond to *spins*, the moves permitted in the game. A *rectangle*  $R$  specifies a rectangular subset of the positions on an  $m \times n$  board, and has dimensions  $i \times j$ , with  $1 \leq i \leq m$  and  $1 \leq j \leq n$ . A spin rotates some rectangle by  $180^\circ$ . It is reasonably clear what this means, but to make it more precise we define a notion of distance that will be useful later.

The *distance*  $\rho(p_1, p_2)$  between positions  $p_1$  and  $p_2$  is measured by applying the  $\ell_1$ -norm to the centers of the corresponding unit squares. Two positions are *adjacent* when they have a single edge in common, equivalently, when the distance between them is 1. For a position  $p$  and a rectangle  $R$ , we use  $\rho(p, R)$  to denote the distance from the center of  $p$  to the center of  $R$  (again using the  $\ell_1$ -norm).

**Definition 2.** The spin about  $R$  is the element of  $\text{Spin}_{m \times n}$  that transposes the tiles in positions  $p_1, p_2 \in R$  if and only if  $\rho(p_1, p_2) = 2\rho(p_1, R) = 2\rho(p_2, R)$ , and then reverses the orientation of each tile in  $R$ .

We say that an element of  $\text{Spin}_{m \times n}$  is a *spin* if it is a spin about some rectangle  $R$ . The proposition below records some useful facts about spins. The proofs are straight-forward, but for the sake of completeness we fill in the details. They can (and probably should) be skipped on a first reading.

**Proposition 1.** Let  $s_1$  and  $s_2$  be spins about rectangles  $R_1$  and  $R_2$  respectively.

- (1)  $s_1$  is its own inverse (as is  $s_2$ ).

- (2)  $s_1 s_2$  is not a spin.
- (3)  $s_1 s_2 = s_2 s_1$  if and only if  $R_1$  and  $R_2$  are disjoint or have a common center.
- (4)  $s_1 s_2 s_1$  is a spin  $s_3$  if and only if either  $s_1$  and  $s_2$  commute or  $R_1$  contains  $R_2$ . The rectangle of  $s_3$  has the same shape as  $R_2$ .

*Proof.* (1) is clear. For (2), suppose  $s_3 = s_1 s_2$  is a spin about some rectangle  $R_3$ . Then  $s_1 \neq s_2$ , and therefore  $R_1 \neq R_2$ , since the identity element is not a spin. Now suppose there exist positions  $p_1 \in R_1 - R_2$  and  $p_2 \in R_2 - R_1$ . Then  $s_3$  moves the tile in position  $p_i$  to the same location that  $s_i$  does, which implies that  $R_i$  and  $R_3$  have a common center, for  $i = 1, 2$ . But then there is a position in  $R_3$  containing this common center (either in its center or along an edge) and  $s_3 = s_1 s_2$  does not change the orientation of the tile in this position, which is a contradiction. Now assume without loss of generality that  $R_1$  properly contains  $R_2$ . Let  $p_1$  and  $p_2$  be corners of  $R_1$  not contained in  $R_2$  (let  $p_1 = p_2$  if  $R_1$  has width or height 1, and  $p_1 \neq p_2$  otherwise). Then  $s_3$  acts on the tiles in positions  $p_1$  and  $p_2$  the same way that  $s_1$  does, and this implies that  $R_3$  and  $R_1$  have a common center and that  $R_1 \subset R_3$ . But  $R_3$  must lie in the union of  $R_1$  and  $R_2$ , so  $R_3 = R_1$  and  $s_3 = s_1$ , but then  $s_2$  is the identity, which is again a contradiction. So  $s_1 s_2$  is not a spin.

We now address (3). For any position  $p$  not in the intersection of  $R_1$  and  $R_2$ , both  $s_1 s_2$  and  $s_2 s_1$  have the same effect on the tile  $t$  in position  $p$ . Now suppose  $p$  is in the intersection of  $R_1$  and  $R_2$ . Then the orientation of  $t$  is preserved by both  $s_1 s_2$  and  $s_2 s_1$ , so we need only consider the position to which  $t$  is moved. The product of two rotations by  $\pi$  is a translation (possibly trivial). Reversing the order of the rotations yields the inverse translation, thus  $t$  is moved to the same position if and only if the translation is trivial, which occurs precisely when  $R_1$  and  $R_2$  have a common center. This proves (3).

For (4), it is clear that if  $s_1$  and  $s_2$  commute then  $s_3 = s_2$  is a spin. Now suppose  $R_1$  contains  $R_2$ . Let  $R_3$  be the inverse image of  $R_2$  under the permutation  $\pi(s_1)$ , and let  $s_3 = s_1 s_2 s_1$ . For tiles in  $R_3$ , the action of  $s_3$  is the product of three rotations by  $\pi$ , which is again a rotation by  $\pi$ , and the center of this rotation is the center of  $R_3$ . Thus  $s_3$  is a spin about  $R_3$ , which has the same shape as  $R_2$ .

To prove the other direction of (4), suppose for the sake of contradiction that  $s_3$  is a spin about some rectangle  $R_3$ , that  $R_1$  and  $R_2$  are not disjoint, do not have a common center, and that  $R_1$  does not contain  $R_2$ . These assumptions guarantee the existence of a position  $p \in R_2 - R_1$  whose image under  $\pi(s_2)$  is in  $R_1$ . The action of  $s_3$  on the tile  $t$  in position  $p$  is the same as  $s_2 s_1$ , which is two rotations by  $\pi$ , hence a translation. But  $R_1$  and  $R_2$  do not have a common center, so this translation is non-trivial and  $s_3$  moves tile  $t$  without changing its orientation, contradicting our assumption that  $s_3$  is a spin.  $\square$

Each spin is uniquely determined by its rectangle  $R$ , thus we may specify a spin in the form  $[p_1, p_2]$ , where  $p_1$  and  $p_2$  identify the positions of the upper left and lower right corners of  $R$  (respectively). For example, on a  $3 \times 3$  board the spin about the  $2 \times 2$  rectangle in the upper right corner is

$$[2, 6] = ((2\ 6)(3\ 5), 011011000).$$

The moves permitted in a game of Spinpossible on an  $m \times n$  board are precisely the set  $\mathcal{S} = \mathcal{S}(m, n)$  of all spins  $[p_1, p_2]$ , where  $1 \leq p_1 \leq p_2 \leq mn$  (we consider variations of the game that place restrictions on  $\mathcal{S}$  in §3).

In mathematical terms, the game works as follows: given an element  $b \in \text{Spin}_{m \times n}$  (the starting board), write  $b^{-1}$  as a product  $s_1 s_2 s_3 \cdots s_k$  of elements in  $\mathcal{S}$ , with  $k$  as small as possible (a *solution*). Applying the spins  $s_1, s_2, \dots, s_k$  to  $b$  then yields the identity (the standard board). In general there will be many solutions to  $b$ , but some boards have a unique solution; this topic is discussed further in §4.

Let  $\mathcal{R}_{i \times j}$  denote the subset of  $\text{Spin}_{m \times n}$  that are spins about an  $i \times j$  rectangle. The set  $\mathcal{R}_{i \times j}$  is necessarily empty if  $i > m$  or  $j > n$ , and we may have  $\mathcal{R}_{i \times j} = \emptyset$  even when  $\mathcal{R}_{j \times i} \neq \emptyset$  (although this can occur only when  $m \neq n$ ). In these notes we shall always consider the sets  $\mathcal{R}_{i \times j}$  and  $\mathcal{R}_{j \times i}$  together, thus we define  $\mathcal{S}_{i \times j} = \mathcal{R}_{i \times j} \cup \mathcal{R}_{j \times i}$ . The set of spins  $\mathcal{S}$  in  $\text{Spin}_{m \times n}$  is then the union of the sets  $\mathcal{S}_{i \times j}$ , each of which we refer to as a *spin type*.

**Proposition 2.** *Assume  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ , and  $m \leq n$ . The following hold:*

- (1)  $|\mathcal{R}_{i \times j}| = (m+1-i)(n+1-j)$ .
- (2)  $|\mathcal{S}_{i \times j}| = \begin{cases} |\mathcal{R}_{i \times j}| + |\mathcal{R}_{j \times i}| & \text{for } i \neq j, \\ |\mathcal{R}_{i \times j}| & \text{for } i = j. \end{cases}$
- (3)  $|\mathcal{S}| = \binom{m+1}{2} \binom{n+1}{2}$ .
- (4) *There are  $\frac{1}{2}m(2n-m+1)$  distinct spin types  $\mathcal{S}_{i \times j}$  in  $\text{Spin}_{m \times n}$ .*

*Proof.* For (1), we note that there are  $(m+1-i)(n+1-j)$  possible locations for the upper left corner of an  $i \times j$  rectangle on an  $m \times n$  board. The formula in (2) is immediate. For (3) we have

$$\sum_{i=1}^m \sum_{j=1}^n (m+1-i)(n+1-j) = \sum_{i=1}^m \sum_{j=1}^n ij = \binom{m+1}{2} \binom{n+1}{2},$$

and for (4) we have

$$\sum_{i=1}^m \sum_{j=i}^n 1 = \sum_{i=1}^m (n+1-i) = m(n+1) - \binom{m+1}{2} = \frac{m(2n-m+1)}{2}.$$

□

It is well known that the symmetric group  $S_N$  is generated by the set of all transpositions (permutations that swap two elements and leave the rest fixed). Slightly less well known is the fact that  $S_N$  is generated by any set of transpositions that form a connected graph, as described in the following lemma.

**Lemma 1.** *Let  $E \subseteq S_N$  be a set of transpositions  $(v_i, v_j)$  acting on a set of vertices  $V = \{v_1, \dots, v_N\}$ . Let  $G$  be the undirected graph on  $V$  with edge set  $E$ . Then  $E$  generates  $S_N$  if and only if  $G$  is connected.*

*Proof.* It suffices to show that  $E$  generates every transposition in  $S_N$ . If the sequence of edges  $(e_1, \dots, e_k)$  is a path from  $v_i$  to  $v_j$  in  $G$ , then the permutation

$$e_1 e_2 \cdots e_{k-2} e_{k-1} e_k e_{k-1} e_{k-2} \cdots e_2 e_1$$

is the transposition  $(v_i, v_j)$ . Let  $H$  be the subgroup of  $S_N$  generated by  $E$ . The  $H$ -orbits of  $V$  correspond to connected components of  $G$ , and  $H$  can achieve any permutation of the vertices in a given component, since it can transpose any pair of vertices connected by a path. Thus  $H = S_N$  if and only if  $G$  is connected. □

**Corollary 1.**  $\mathcal{S}_{1 \times 1} \cup \mathcal{S}_{1 \times 2}$  generates  $\text{Spin}_{m \times n}$ .

*Proof.* The set  $\mathcal{S}_{1 \times 2}$  consists of transpositions that form a connected graph whose vertices are the positions on an  $m \times n$  board with edges between adjacent positions. It follows from Lemma 1 that, given any element  $(\alpha, \mathbf{u})$  in  $\text{Spin}_{m \times n}$ , there is a vector  $\mathbf{v} \in B_{mn}$  for which we can construct  $(\alpha, \mathbf{v})$  as a product of elements in  $\mathcal{S}_{1 \times 2}$ . By applying appropriate elements of  $\mathcal{S}_{1 \times 1}$  to  $(\alpha, \mathbf{v})$  we can obtain  $(\alpha, \mathbf{u})$ .  $\square$

The corollary implies that every starting board in the Spinpossible game has a solution. We now give an upper bound on the length of any solution.

**Theorem 1.** *Every element of  $\text{Spin}_{m \times n}$  can be expressed as a product of at most  $3mn - (m + n)$  spins.*

*Proof.* Let  $(\alpha, \mathbf{u})$  be an element of  $\text{Spin}_{m \times n}$ . For any  $\mathbf{v} \in V_N$  we may write  $(\alpha, \mathbf{u})$  as  $(\alpha, \mathbf{v})(\iota, \mathbf{u} + \mathbf{v})$ , where  $\iota$  denotes the trivial permutation. It is clear that we can express  $(\iota, \mathbf{u} + \mathbf{v})$  as the product of at most  $mn$  elements in  $\mathcal{S}_{1 \times 1}$ . Thus it suffices to show that we can construct an element of the form  $(\alpha, \mathbf{v})$ , for some  $\mathbf{v} \in V_N$ , as a product of at most  $2mn - (m + n)$  spins. Since we may use any  $\mathbf{v}$  we like, we now ignore the orientation of tiles and focus on the permutation  $\alpha$ . Rather than constructing  $\alpha$ , we shall construct  $\alpha^{-1}$  (which is equivalent, since  $\alpha$  is arbitrary).

We now proceed by induction on  $N$  to show that we can construct  $\alpha^{-1}$  using at most  $2mn - (m + n)$  spins. For  $N = 1$  we necessarily have  $\alpha^{-1} = \iota$ , which is the product of  $0 = 2mn - (m + n)$  spins. For  $N > 1$ , assume without loss of generality that  $m \leq n$  (interchange the role of rows and columns in what follows if not). We first use the spin about the rectangle  $[1, \alpha(1)]$  to restore tile 1 to its correct position in the upper left corner. Now let  $i$  and  $j$  be the vertical and horizontal distances, respectively, between positions  $n + 1$  and  $\alpha(n + 1)$ , so that  $\rho(n + 1, \alpha(n + 1)) = i + j$ . To move tile  $n + 1$  to position  $n + 1$  (the second row of the leftmost column) we first apply an element of  $\mathcal{S}_{(i+1) \times 1}$  to move tile  $n + 1$  to the correct row, and then apply an element of  $\mathcal{S}_{1 \times (j+1)}$  to move tile  $n + 1$  to the correct column (we can omit spins in  $\mathcal{S}_{1 \times 1}$ , which arise when  $i$  or  $j$  is zero). Neither of these spins affects position 1. In a similar fashion, we can successively move each tile  $kn + 1$  for  $2 \leq k < m$  from position  $\alpha(kn + 1)$  to position  $kn + 1$  using at most two spins per tile, without disturbing any of the tiles in positions  $jn + 1$  for  $j < k$ .

The total number of spins used to correctly position all the tiles in the leftmost column is  $2m - 1$ . By the inductive hypothesis, we can correctly position the remaining tiles in the  $(m - 1) \times n$  board obtained by ignoring the leftmost column using at most  $2(m - 1)n - (m - 1 + n)$  spins. The total number of spins used is

$$2m - 1 + 2(m - 1)n - (m - 1 + n) = 2mn + m - 3n - 1,$$

and since  $m \leq n$  this is less than  $2mn - (m + n)$ .  $\square$

The upper bound in Theorem 1 can be improved for  $mn > 1$ . A more detailed analysis of the case  $\text{Spin}_{3 \times 3}$  shows that one can move every tile to its correct position using at most 9 spins<sup>1</sup>, and then orient every tile correctly using at most 7 spins, yielding an upper bound of 16, versus the bound of 21 given by Theorem 1. We also note that the leading constant 3 is not the best possible: for  $m, n \geq 3$  the technique used to orient tiles in the  $3 \times 3$  case can be generalized to achieve  $25/9$ .

Let  $k(m, n)$  denote the maximum length of a solution to a board in  $\text{Spin}_{m \times n}$ . Theorem 1 gives an upper bound on  $k(m, n)$ . We now prove a lower bound.

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<sup>1</sup>It is known that 7 spins always suffice, and are sometimes necessary.

**Theorem 2.** *Assume  $N = mn > 1$ . Then*

$$k(m, n) \geq \frac{\ln(2^N N!)}{\ln\left(\binom{m+1}{2}\binom{n+1}{2} + 1\right)}.$$

*This implies the bound*

$$k(m, n) \geq \frac{1}{2}mn - \frac{(1 - \ln 2)}{2} \cdot \frac{mn}{\ln mn} + \frac{1}{4}.$$

*Proof.* Let  $c = |\mathcal{S}| + 1$ . The number of distinct expressions of the form  $s_1 \cdots s_j$  with  $s_1, \dots, s_j \in \mathcal{S}$  and  $j \leq k$  is at most  $c^k$ . Not all of these expressions yield distinct elements of  $\text{Spin}_{m \times n}$ , but in any case it is clear that they correspond to at most  $c^k$  distinct elements of  $\text{Spin}_{m \times n}$ . The cardinality of  $\text{Spin}_{m \times n}$  is  $2^N N!$ , thus in order to express every element of  $\text{Spin}_{m \times n}$  as a product of at most  $k$  spins we must have

$$(1) \quad c^k \geq 2^N N!$$

From Proposition 2 we have  $c = \binom{m+1}{2}\binom{n+1}{2} + 1$ . Taking logarithms in (1) and dividing by  $\ln c$  yields the first bound for  $k(m, n)$ .

For the second bound, we note that  $N^2 > c$  for all  $N = mn > 1$ , thus we can replace the LHS of (1) by  $N^{2k}$ . By bounding the error term in Stirling's approximation one can show that

$$\ln N! \geq N \ln N - N + \frac{1}{2} \ln N,$$

for all  $N \geq 1$ . Applying  $N^{2k} > c^k$  and taking logarithms in (1) yields

$$2k \ln N \geq N \ln N - (1 - \ln 2)N + \frac{1}{2} \ln N.$$

Dividing by  $2 \ln N$  gives

$$k \geq \frac{1}{2}N - \frac{1 - \ln 2}{2} \cdot \frac{N}{\ln N} + \frac{1}{4},$$

which proves the second bound.  $\square$

For  $m = n = 3$ , Theorem 2 give the lower bound  $k(3, 3) \geq 6$ , which is not far below the known value  $k(3, 3) = 9$ . Asymptotically, we have the following corollary.

**Corollary 2.** *The asymptotic growth of  $k(m, n)$  is linear in  $N = mn$ . More precisely, for every  $\epsilon > 0$  there is an  $N_0$  such that*

$$\left(\frac{1}{2} + \epsilon\right)N < k(m, n) \leq 3N$$

*for all  $N > N_0$ .*

Recall that for a group  $G$  generated by a set  $S$ , the Cayley graph  $\text{Cay}(G, S)$  is the graph with vertex set  $G$  and edge  $(g, h)$  labelled by  $s$  whenever  $sg = h$ , where  $s \in S$  and  $g, h \in G$ . A solution to a board  $b \in \text{Spin}_{m \times n}$  corresponds to a shortest path from  $b$  to the identity in the graph  $\text{Cay}(\text{Spin}_{m \times n}, \mathcal{S})$ . The quantity  $k(m, n)$  is the diameter of this graph.

## 3. RESTRICTED SPIN SETS

Spinpossible includes variations of the standard game that place restrictions on the types of spins that are allowed. For example, the “no singles/doubles” puzzle levels prohibit the use of spins in  $\mathcal{S}_{1 \times 1}$  and  $\mathcal{S}_{1 \times 2}$ . This raises the question of whether it is still possible to solve every board under such a restriction. More generally, we may ask: which subsets of the full set of spins  $\mathcal{S} = \mathcal{S}(m, n)$  generate  $\text{Spin}_{m \times n}$ ?

We begin by defining three subsets of  $\mathcal{S}$  that cannot generate  $\text{Spin}_{m \times n}$  when  $mn > 1$ , using three different notions of parity.

- (1) The *even area* spins  $\mathcal{S}^a$  are the spins whose rectangles have even area.  $\mathcal{S}^a$  is the union of the  $\mathcal{S}_{i \times j}$  for which  $ij \equiv 0 \pmod{2}$ .
- (2) The *even permutation* spins  $\mathcal{S}^p$  are the spins that contain an even number of transpositions.  $\mathcal{S}^p$  is the union of the  $\mathcal{S}_{i \times j}$  for which  $ij \equiv 0 \text{ or } 1 \pmod{4}$ .
- (3) The *even distance* spins  $\mathcal{S}^d$  are the spins that transpose positions at even distances.  $\mathcal{S}^d$  is the union of the  $\mathcal{S}_{i \times j}$  for which  $i + j \equiv 0 \pmod{2}$ .

We now consider the corresponding subgroups of  $\text{Spin}_{m \times n}$ . In these definitions  $\alpha$  is a permutation in  $S_N$ ,  $\mathbf{v}$  is a vector in  $V_N$ , and  $\text{wt}(\mathbf{v})$  denotes the Hamming weight of  $\mathbf{v}$  (the number of 1s it contains). The group  $A_N$  is the alternating group in  $S_N$ , and we define the permutation group  $D_N \cong S_{\lceil N/2 \rceil} \times S_{\lfloor N/2 \rfloor}$  as follows:

$$D_N = \{\alpha : \rho(i, \alpha(i)) \equiv 0 \pmod{2} \text{ for } 1 \leq i \leq N\}.$$

Here  $i$  and  $\alpha(i)$  identify positions on an  $m \times n$  board and  $\rho(i, \alpha(i))$  is the  $\ell_1$ -distance.

The subgroups  $\text{Spin}_{m \times n}^*$ , where  $*$  is  $a$ ,  $p$ , or  $d$ , are defined as follows:

- (1)  $\text{Spin}_{m \times n}^a = \{(\alpha, \mathbf{v}) : \text{wt}(\mathbf{v}) \equiv 0 \pmod{2}\} \quad (\text{index } 2).$
- (2)  $\text{Spin}_{m \times n}^p = \{(\alpha, \mathbf{v}) : \alpha \in A_N\} \quad (\text{index } 2).$
- (3)  $\text{Spin}_{m \times n}^d = \{(\alpha, \mathbf{v}) : \alpha \in D_N\} \quad (\text{index } \binom{N}{\lfloor N/2 \rfloor}).$

It is not necessarily the case that  $\mathcal{S}^*$  generates  $\text{Spin}_{m \times n}^*$ , but we always have  $\mathcal{S}^* = \mathcal{S} \cap \text{Spin}_{m \times n}^*$ . In particular, it is clear that  $\langle \mathcal{S}^* \rangle \subseteq \text{Spin}_{m \times n}^*$ . The following propositions give some conditions under which equality holds.

**Proposition 3.** *Assume  $m, n \geq 2$  and  $mn > 4$ . Then  $\mathcal{S}_{1 \times 2} \cup \mathcal{S}_{2 \times 2}$  (and therefore  $\mathcal{S}^a$ ) generates  $\text{Spin}_{m \times n}^a$ .*

*Proof.* Let  $G = \langle \mathcal{S}_{1 \times 2} \cup \mathcal{S}_{2 \times 2} \rangle$ , and let  $\hat{\pi}$  denote the restriction of  $\pi$  to  $G$ . The fact that  $G$  contains  $\mathcal{S}_{1 \times 2}$  implies that  $\pi(G) = S_N$ , by Lemma 1. It thus suffices to show that the kernel of  $\hat{\pi}$  has index 2 in  $\ker \pi \cong V_N$ .

The following product of spins in  $G$  transposes the tiles in positions 1 and 2:

$$(2) \quad [2, 3][1, n+1][1, n+2][2, 3][1, n+2][1, n+1][2, 3] = ((1 \ 2), \mathbf{0})$$

We can transform the identity above by applying any square-preserving isometry of  $\mathbb{Z}^2$  (the group generated by unit translations and reflections about the lines  $y = 0$  and  $y = x$ ). Such a transformation may change the location and/or orientation of the rectangles identifying the spins that appear in the product, but it does not change their spin type (the set  $\mathcal{S}_{i \times j}$  to which they belong). This allows us to transpose any pair of adjacent tiles on the  $m \times n$  board using a product of spins in  $G$ . It follows from Lemma 1 that  $G$  contains the subgroup  $H = \{(\alpha, \mathbf{0}) : \alpha \in S_N\}$ .

For each even integer  $w$  from 0 to  $N$ , we can construct some  $g_w = (\beta, \mathbf{v})$  with  $\text{wt}(\mathbf{v}) = w$ , as a product of elements in  $\mathcal{S}_{1 \times 2}$ . The coset  $g_w H \subset G$  then contains elements of the form  $(\beta, \mathbf{v})$  for every vector  $\mathbf{v}$  with  $\text{wt}(\mathbf{v}) = w$ . Multiplying each



$(\beta, \mathbf{v})$  on the left by  $(\beta^{-1}, \mathbf{0})$ , we see that  $G$  contains elements  $(\iota, \mathbf{v})$  for every even weight vector  $\mathbf{v}$ . Therefore  $\ker \hat{\pi}$  has index 2 in  $\ker \pi$ .  $\square$

We note that  $\mathcal{S}^a$  does not generate  $\text{Spin}_{m \times n}^a$  when  $m = n = 2$ , nor when exactly one of  $m$  or  $n$  is 1.

**Proposition 4.** *Assume  $mn \neq 4$ . Then  $\mathcal{S}_{1 \times 1} \cup \mathcal{S}_{2 \times 2} \cup \mathcal{S}_{1 \times 3}$  (and therefore  $\mathcal{S}^d$ ) generates  $\text{Spin}_{m \times n}^d$ .*

*Proof.* Let  $G = \langle \mathcal{S}_{1 \times 1} \cup \mathcal{S}_{2 \times 2} \cup \mathcal{S}_{1 \times 3} \rangle$ . Then  $G$  contains  $\ker \pi = \langle \mathcal{S}_{1 \times 1} \rangle \cong V_N$ . It remains to show that  $\pi(G) = D_N$ .

Assume for the moment that  $m \geq 2$  and  $n \geq 3$ . The following product of spins in  $G$  transpose the tiles in positions 1 and  $n+2$ :

$$(3) \quad [2, n+3][1, 3][2, n+3][n+3, n+3] = ((1 \ n+2), \mathbf{0})$$

As in the proof of Proposition 3, we may transform this identity by applying any square-preserving isometry of  $\mathbb{Z}^2$ . Thus we can transpose any pair of tiles that share exactly one common vertex (i.e., that are “diagonally adjacent”), and we can also handle the case  $m \geq 3$  and  $n \geq 2$ . It then follows from Lemma 1 that these transpositions generate  $D_N$ .

We now consider the case where  $m$  or  $n$  is 1. If  $mn \leq 2$  the proposition clearly holds (we only need spins in  $\mathcal{S}_{1 \times 1}$ ), so assume without loss of generality that  $m = 1$  and  $n \geq 3$ . We now replace (3) with

$$[1, 1][1, 3][2, 2][1, 1] = ((1 \ 3), \mathbf{0}),$$

and apply the same argument.  $\square$

It is easy to check that when  $mn = 4$  the set  $\mathcal{S}^d$  does not generate  $\text{Spin}_{m \times n}^d$ .

We leave open the question of when  $\mathcal{S}^p$  generates  $\text{Spin}_{m \times n}^p$ , but for  $\text{Spin}_{3 \times 3}$  we note that  $\mathcal{S}^p \subset \mathcal{S}^a$  (see below), thus  $\mathcal{S}^p$  does not generate  $\text{Spin}_{m \times n}$  in this case.

For reference, we list the spin types  $\mathcal{S}_{i \times j}$  contained in  $\mathcal{S}^a$ ,  $\mathcal{S}^p$ , and  $\mathcal{S}^d$  for all  $i, j \leq 3 \leq m, n$ :

- (1)  $\mathcal{S}_{1 \times 2}, \mathcal{S}_{2 \times 2}, \mathcal{S}_{2 \times 3} \subset \mathcal{S}^a$ .
- (2)  $\mathcal{S}_{2 \times 2}, \mathcal{S}_{3 \times 3} \subset \mathcal{S}^p$ .
- (3)  $\mathcal{S}_{1 \times 1}, \mathcal{S}_{1 \times 3}, \mathcal{S}_{2 \times 2}, \mathcal{S}_{3 \times 3} \subset \mathcal{S}^d$ .

To simplify our analysis of the subsets of  $\mathcal{S}$  that generate  $\text{Spin}_{m \times n}$ , we introduce an equivalence relation on spin types.

**Definition 3.** *Two spin types  $\mathcal{S}_{i \times j}$  and  $\mathcal{S}_{i' \times j'}$  are equivalent, denoted  $\mathcal{S}_{i \times j} \sim \mathcal{S}_{i' \times j'}$ , whenever  $\langle \mathcal{S}_{i \times j} \rangle = \langle \mathcal{S}_{i' \times j'} \rangle$ .*

**Proposition 5.** *Let  $m, n \geq 3$ . For  $1 \leq i, i', j, j' \leq 3$  there is exactly one non-trivial equivalence of spin types  $\mathcal{S}_{i \times j} \sim \mathcal{S}_{i' \times j'}$ , namely,  $\mathcal{S}_{1 \times 2} \sim \mathcal{S}_{2 \times 3}$*

*Proof.* If a particular spin type is contained in  $\mathcal{S}^*$  (where  $*$  is  $a$ ,  $p$ , or  $d$ ), then so is every equivalent spin type. Examining the list of spin types for  $\mathcal{S}^*$ , we can use this criterion to rule out all but two possible equivalences among the 6 spin types  $\mathcal{S}_{i \times j}$  with  $1 \leq i, j \leq 3$ . The first is the pair  $\mathcal{S}_{1 \times 1}$  and  $\mathcal{S}_{3 \times 3}$ , but these cannot be equivalent because  $\langle \mathcal{S}_{1 \times 1} \rangle$  lies in  $\ker \pi_1 \cong V_N$  but  $\mathcal{S}_{3 \times 3}$  does not. The second is the pair  $\mathcal{S}_{1 \times 2}$  and  $\mathcal{S}_{2 \times 3}$ , which we now show are equivalent.

For simplicity we shall write spins in terms of rectangles with coordinates on a  $3 \times 3$  board, but these can be generalized to an  $m \times n$  board by replacing positions



4, 5, 6, 7, 8 and 9 with positions  $n+1$ ,  $n+2$ ,  $n+3$ ,  $2n+1$ ,  $2n+2$ , and  $2n+3$ , respectively. We can write the spin  $[1, 6]$  as a product of spins in  $\mathcal{S}_{1 \times 2}$  as follows:

$$[1, 6] = [2, 5][2, 3][4, 5][5, 6][1, 2][4, 5][2, 3][3, 6][1, 4].$$

As in the proofs of Propositions 3 and 4, we can transform this identity via a square-preserving isometry of  $\mathbb{Z}^2$  to express any spin in  $\mathcal{S}_{2 \times 3}$  as a product of spins in  $\mathcal{S}_{1 \times 2}$ . Thus  $\langle \mathcal{S}_{2 \times 3} \rangle \subset \langle \mathcal{S}_{1 \times 2} \rangle$ . For the other inclusion, we may write the spins  $[1, 2]$  and  $[4, 5]$  as products of spins in  $\mathcal{S}_{2 \times 3}$  as follows:

$$[1, 2] = [1, 6][4, 9][1, 8][4, 9][1, 8][1, 6][1, 8][4, 9][1, 8][1, 6][1, 8][4, 9][1, 8][1, 6][1, 8],$$

$$[4, 5] = [1, 6][2, 9][1, 6][2, 9][4, 9][2, 9][1, 6][2, 9][4, 9][2, 9][1, 6][2, 9][4, 9][2, 9][4, 9][2, 9][4, 9]$$

By transforming one of these two identities with a suitable isometry we can express any spin in  $\mathcal{S}_{1 \times 2}$  as a product of spins in  $\mathcal{S}_{2 \times 3}$ . Thus  $\langle \mathcal{S}_{1 \times 2} \rangle \subset \langle \mathcal{S}_{2 \times 3} \rangle$ .  $\square$

We are now ready to prove our main theorem, which completely determines the combinations of spin types that generate  $\text{Spin}_{3 \times 3}$ .

**Theorem 3.** *Assume that  $m, n \geq 3$ . Let  $\mathcal{T}$  be a union of spin types  $\mathcal{S}_{i \times j}$ , where  $1 \leq i, j \leq 3$ . For  $\mathcal{T}$  to generate  $\text{Spin}_{m \times n}$ , it is sufficient for  $\mathcal{T}$  to contain one of the following six sets:*

$$\begin{aligned} \mathcal{S}_{1 \times 2} \cup \mathcal{S}_{1 \times 1}, \quad \mathcal{S}_{1 \times 2} \cup \mathcal{S}_{1 \times 3} \quad & \mathcal{S}_{1 \times 2} \cup \mathcal{S}_{2 \times 2} \cup \mathcal{S}_{3 \times 3}, \\ \mathcal{S}_{2 \times 3} \cup \mathcal{S}_{1 \times 1}, \quad \mathcal{S}_{2 \times 3} \cup \mathcal{S}_{1 \times 3} \quad & \mathcal{S}_{2 \times 3} \cup \mathcal{S}_{2 \times 2} \cup \mathcal{S}_{3 \times 3}. \end{aligned}$$

When  $m = n = 3$ , this condition is also necessary.

*Proof.* We first prove sufficiency. By Proposition 5,  $\mathcal{S}_{1 \times 2} \sim \mathcal{S}_{2 \times 3}$ , so it is enough to prove that each of the first three sets listed in the theorem generates  $\text{Spin}_{m \times n}$ . As above, we specify spins using coordinates on a  $3 \times 3$  board, but these can coordinates can be transferred to an  $m \times n$  board as noted in the proof of Proposition 5.

By Corollary 1, the set  $\mathcal{S}_{1 \times 1} \cup \mathcal{S}_{1 \times 2}$  generates  $\text{Spin}_{m \times n}$ . For  $\mathcal{S}_{1 \times 2} \cup \mathcal{S}_{1 \times 3}$ , it is enough to show that  $\mathcal{S}_{1 \times 1} \subset \langle \mathcal{S}_{1 \times 2} \cup \mathcal{S}_{1 \times 3} \rangle$ . We note that each element of  $\mathcal{S}_{1 \times 1}$  has the form  $s_i = (\iota, \mathbf{e}_i)$ , where  $\mathbf{e}_i$  is the weight 1 vector in  $V_N$  with the  $i$ th bit set. If  $(\alpha, \mathbf{u})$  is any element of  $\text{Spin}_{m \times n}$  with  $\alpha(1) = i$ , then we have

$$\begin{aligned} (\alpha, \mathbf{u})^{-1}(\iota, \mathbf{e}_1)(\alpha, \mathbf{u}) &= (\alpha^{-1}, \mathbf{u}^{\alpha^{-1}})(\iota, \mathbf{e}_1)(\alpha, \mathbf{u}) \\ &= (\alpha^{-1}, \mathbf{u}^{\alpha^{-1}} + \mathbf{e}_1)(\alpha, \mathbf{u}) \\ &= (\iota, \mathbf{e}_1^\alpha) = (\iota, \mathbf{e}_i). \end{aligned}$$

Since  $\pi(\langle \mathcal{S}_{1 \times 2} \rangle) = S_N$ , by Lemma 1, we can generate a suitable  $(\alpha, \mathbf{u})$  for each  $i$  from 1 to  $N$ . Thus it is enough to show how to express the spin  $[1, 1]$  as a product of spins in  $\mathcal{S}_{1 \times 2} \cup \mathcal{S}_{1 \times 3}$ :

$$[1, 1] = [1, 2][1, 4][1, 3][1, 4][1, 2][4, 6][3, 6][4, 6].$$

The same arguments apply to the third set  $\mathcal{S}_{1 \times 2} \cup \mathcal{S}_{2 \times 2} \cup \mathcal{S}_{3 \times 3}$ , thus it suffices to note that:

$$[1, 1] = [2, 3][1, 5][1, 2][3, 6][4, 7][1, 5][2, 3][5, 8][1, 9][5, 9][1, 5][5, 8][8, 9].$$

We now prove the necessity of the condition in the theorem, under the assumption  $m = n = 3$ . The set  $\mathcal{T}$  is the union of some subset of the six spin types

$$\mathcal{U} = \{\mathcal{S}_{1 \times 1}, \mathcal{S}_{1 \times 2}, \mathcal{S}_{1 \times 3}, \mathcal{S}_{2 \times 2}, \mathcal{S}_{2 \times 3}, \mathcal{S}_{3 \times 3}\}.$$

Of the 64 subsets of  $\mathcal{U}$ , one finds that 22 of them have unions that are contained in  $\mathcal{S}^a$  or  $\mathcal{S}^d$ , thus  $\mathcal{T}$  cannot be the union of any of these 22 subsets. Conversely, one finds that 39 of the remaining 42 subsets of  $\mathcal{U}$  have unions containing one of the 6 sets listed in the proposition. The 3 remaining subsets of  $\mathcal{U}$  all have unions contained in  $\mathcal{S}_{1 \times 2} \cup \mathcal{S}_{2 \times 3} \cup \mathcal{S}_{3 \times 3}$ , which we now argue does not generate  $\text{Spin}_{m \times n}$ .

Since  $\mathcal{S}_{1 \times 2} \sim \mathcal{S}_{2 \times 3}$ , it is enough to show that  $\mathcal{S}_{1 \times 2} \cup \mathcal{S}_{3 \times 3}$  does not generate  $\text{Spin}_{m \times n}$ . By Lemma 6 below, any product of elements in  $\mathcal{S}_{1 \times 2} \cup \mathcal{S}_{3 \times 3}$  is equivalent to a product in which the unique element of  $\mathcal{S}_{3 \times 3}$  appears only once, in the rightmost position. It follows that the cardinality of  $\langle \mathcal{S}_{1 \times 2} \cup \mathcal{S}_{3 \times 3} \rangle$  is at most (in fact, exactly) twice that of  $\langle \mathcal{S}_{1 \times 2} \rangle$ . But by Lemma 2 below, the subgroup  $\langle \mathcal{S}_{1 \times 2} \rangle$  has trivial intersection with  $\ker \pi$  and thus has index  $2^9 = 512$  in  $\text{Spin}_{3 \times 3}$ . So  $\langle \mathcal{S}_{1 \times 2} \cup \mathcal{S}_{3 \times 3} \rangle$  is a proper subgroup of  $\text{Spin}_{3 \times 3}$ .  $\square$

**Lemma 2.** *The restriction of the projection map  $\pi: \text{Spin}_{m \times n} \rightarrow S_N$  to the group  $G = \langle \mathcal{S}_{1 \times 2} \rangle$  is an isomorphism from  $G$  to  $S_N$ .*

*Proof.* Let  $\hat{\pi}: G \rightarrow S_N$  be the restriction of  $\pi$  to  $G$ . The fact that  $\hat{\pi}$  is surjective follows from Lemma 1, so we only need to show that  $\hat{\pi}$  is injective. Let  $h$  be any element of the kernel of  $\hat{\pi}$ . Then  $h = s_1 \cdots s_k$  is a product of spins in  $\mathcal{S}_{1 \times 2}$ , and  $h$  fixes the position of every tile on the  $m \times n$  board. We will show that  $h$  also fixes the orientation of every tile, and therefore  $h$  is the identity.

Consider tile  $t$  in position  $t$  on the standard board  $b$ . If we apply  $h$  to  $b$ , each spin  $s_i$  potentially moves the tile  $t$ , but if it does, it always moves  $t$  to an adjacent position on the board, since  $s_i \in \mathcal{S}_{1 \times 2}$ . Thus  $t$  is moved along some path on the  $m \times n$  board (possibly trivial) that must eventually return  $t$  to its original position. The length of this path is necessarily an even integer, therefore  $t$  is also returned to its original orientation.  $\square$

Let  $\mathcal{T}$  be a subset of the spins in  $\mathcal{S}$ . Generalizing our definition of  $k(m, n)$ , we define  $k(m, n, \mathcal{T})$  as the diameter of the Cayley graph  $\text{Cay}(\text{Spin}_{m \times n}, \mathcal{T})$ , and consider upper and lower bounds for  $k(m, n, \mathcal{T})$ . To do so, we introduce a notion of *weight* for a spin, defined the total distance covered by all the tiles it moves.

**Definition 4.** *The weight of a rectangle  $R$  is  $\text{wt}(R) = 2 \sum_{p \in R} \rho(p, R)$ , and the weight of a spin  $s$  about  $R$  is  $\text{wt}(s) = \text{wt}(R)$ .*

We may denote the weight of an  $i \times j$  rectangle  $R$  by  $w(i, j)$ , since it depends only on the dimensions of  $R$ , not its location.

**Lemma 3.** *Let  $\varepsilon: \mathbb{Z} \rightarrow \{0, 1\}$  be the parity map. Then*

$$w(m, n) = \frac{1}{2}(mn^2 + nm^2 - \varepsilon(m)n - \varepsilon(n)m).$$

*Proof.* When  $m$  and  $n$  are both even we have

$$w(m, n) = 4 \cdot 2 \left( \sum_{i=1}^{m/2} \sum_{j=1}^{n/2} (i + j - 1) \right) = \frac{1}{2}(mn^2 + nm^2).$$

When  $m$  and  $n$  are both odd we have

$$\begin{aligned} w(m, n) &= 4 \cdot 2 \left( \sum_{i=1}^{\frac{m-1}{2}} \sum_{j=1}^{\frac{n-1}{2}} (i+j) \right) + 2 \cdot 2 \left( \sum_{i=1}^{\frac{m-1}{2}} i + \sum_{j=1}^{\frac{n-1}{2}} j \right) \\ &= \frac{1}{2} (mn^2 + nm^2 - m - n). \end{aligned}$$

The cases where  $m$  and  $n$  have opposite parity are similar and left to the reader.  $\square$

**Lemma 4.** *Let  $\mathcal{T}$  be any set of spins in  $\text{Spin}_{m \times n}$ . Then*

$$k(m, n, \mathcal{T}) \geq \frac{w(m, n)}{\max\{\text{wt}(s) : s \in \mathcal{T}\}}$$

for all  $m, n \geq 1$ .

*Proof.* Let  $b \in \mathcal{S}_{m \times n}$ , with  $\text{wt}(b) = w(m, n)$ . If  $s_1 \cdots s_k$  is a product of spins in  $\mathcal{T}$  equivalent to  $b$ , then  $w(m, n) \leq \sum \text{wt}(s_i) \leq kw_{\max}$ . The lemma follows.  $\square$

**Lemma 5.** *Every element of  $\text{Spin}_{m \times n}$  can be expressed as the product of at most  $w(m, n) + mn$  spins in  $\mathcal{S}_{1 \times 1} \cup \mathcal{S}_{1 \times 2}$ .*

*Proof.* Let  $b \in \text{Spin}_{m \times n}$ . We will construct  $b^{-1} = (\alpha, \mathbf{u})$  by constructing an element  $(\alpha, \mathbf{v})$  as a product of at most  $w(m, n)$  spins in  $\mathcal{S}_{1 \times 2}$ , to which we may then apply at most  $mn$  spins in  $\mathcal{S}_{1 \times 1}$  to obtain  $(\alpha, \mathbf{v})$ .

Let  $d = m + n - 2$ . Then  $d$  is the maximum ( $\ell_1$ ) distance between any position and the center of the  $m \times n$  rectangle  $R$  containing all the positions on the board. For each position  $i$  at distance  $d$  from the center (the 4 corners when  $mn > 1$ ), we can move tile  $i$  to position  $i$  using at most  $2d$  spins in  $\mathcal{S}_{1 \times 2}$ . Next we place the correct tiles in positions at distance  $d - 1$  from the center, and each of these tiles can currently lie at most  $d - 1$  positions away from the center (since the distance  $d$  positions are already filled with the correct tiles), thus we use at most  $2(d - 1)$  spins in  $\mathcal{S}_{1 \times 2}$  to place the correct tiles in the positions at distance  $d - 1$  from the center. Note that we can do this by moving each tile along a path that does not disturb any tiles that have already been placed. Continuing in this fashion, we use at most  $2\rho(p, R)$  spins to place the correct tile in position  $p$ , and the total number of spins is at most  $\text{wt}(R) = w(m, n)$ .  $\square$

**Corollary 3.** *For all  $m, n \geq 1$  let  $\mathcal{T} = \mathcal{T}(m, n)$  be a set of spins with weight bounded by some constant  $W$ . Then as  $N = mn \rightarrow \infty$  we have the asymptotic bound  $k(m, n, \mathcal{T}) = \Theta(mn^2 + nm^2)$ . More precisely, for every  $\epsilon > 0$  there is an  $N_0$  such that*

$$\left( \frac{1}{2W} + \epsilon \right) (mn^2 + nm^2) < k(m, n, \mathcal{T}) < (1 + \epsilon)(mn^2 + nm^2),$$

for all  $N > N_0$ .

For  $m = n$  this gives a  $\Theta(N^{1.5})$  bound, which may be contrasted with the  $\Theta(N)$  bound of Corollary 2, where the weight of the spins was unrestricted. We note that in the case of  $\text{Spin}_{3 \times 3}$  and  $\mathcal{T} = \mathcal{S}_{1 \times 1} \cup \mathcal{S}_{1 \times 2}$ , Lemmas 4 and 5 give the bounds  $12 < k(3, 3, \mathcal{T}) < 33$ , compared to the actual value  $k(3, 3, \mathcal{T}) = 25$ .

## 4. UNIQUE SOLUTIONS

Certain elements of  $\text{Spin}_{m \times n}$  are distinguished by the fact that they have a unique solution (a unique shortest expression as a product of spins). This is clearly the case, for example, when  $b \in \mathcal{S}$ . There are many less trivial examples, some 2,203,401 of them in  $\text{Spin}_{3 \times 3}$ . These include what appear to be the most difficult puzzles in the game, some of which are featured in separate puzzle levels designated as “uniques”. While these can be quite challenging, knowing that the solution is unique can be an aid to solving such a puzzle.

We begin with a lemma used in the proof of Theorem 3, which also allows us to rule out many possible candidates for a unique solution.

**Lemma 6.** *Let  $b = s_1 \cdots s_i \cdots s_k$  be a product of spins in  $\text{Spin}_{m \times n}$ , with  $i < k$  and  $s_i \in \mathcal{S}_{1 \times 1} \cup \mathcal{S}_{m \times n}$ . If  $s_i \in \mathcal{S}_{1 \times 1}$ , then  $b$  can be written as  $b = s_1 \cdots s_{i-1} s_{i+1} \cdots s_k t_i$  with  $t_i \in \mathcal{S}_{1 \times 1}$ . If  $s_i \in \mathcal{S}_{m \times n}$ , then  $b$  can be written as  $b = s_i \cdots s_{i-1} t_{i+1} \cdots t_k s_i$ , with each  $t_j$  a spin of the same type as  $s_j$ , for  $i \leq j \leq k$ .*

*Proof.* We first suppose that  $s_i \in \mathcal{S}_{1 \times 1}$ . Then the rectangle  $R_i$  of  $s_i$  contains just a single position. Let  $R_{i+1}$  be the rectangle of  $s_{i+1}$ . If  $R_i$  is contained in  $R_{i+1}$ , then by Proposition 1, we have  $s_{i+1} s_i s_{i+1} = t$  with  $t \in \mathcal{S}_{1 \times 1}$ . Multiplying on the left by  $s_{i+1}$ , we have  $s_i s_{i+1} = s_{i+1} t_i$ , allowing us to “shift” the spin  $s_i$  to the right, potentially changing the location of its rectangle but not its type. If  $R_i$  is not contained in  $R_{i+1}$  then  $R_i$  and  $R_{i+1}$  are disjoint and we simply let  $t = s_i$ , since then  $t$  and  $s_{i+1}$  commute. Applying the same procedure to  $s_{i+2}, \dots, s_k$ , we eventually obtain a product  $b = s_1 \cdots s_{i-1} s_{i+1} \cdots s_k t_i$  of the desired form (using a potentially different  $t$  at each step).

We now suppose that  $s_i \in \mathcal{S}_{m \times n}$ . Then the rectangle  $R_i$  of  $s_i$  covers the entire  $m \times n$  board. Let  $R_{i+1}$  be the rectangle of  $s_{i+1}$ , which is necessarily contained in  $R_i$ . We then have  $s_i s_{i+1} s_i = t_{i+1}$ , where  $t_{i+1}$  is a spin of the same type as  $s_{i+1}$ , and therefore  $s_i s_{i+1} = t_{i+1} s_i$ . We may proceed in the same fashion to compute  $t_{i+2}, \dots, t_k$ , eventually obtaining the desired product  $b = s_i \cdots s_{i-1} t_{i+1} \cdots t_k s_i$ .  $\square$

**Proposition 6.** *Suppose that  $s_1 \cdots s_k$  is the unique solution to a board  $b$  in  $\text{Spin}_{m \times n}$ . Then the following hold:*

- (1) *None of the  $s_i$  are contained in  $\mathcal{S}_{1 \times 1}$  or  $\mathcal{S}_{m \times n}$ .*
- (2) *Consecutive pairs  $s_i$  and  $s_{i+1}$  have rectangles  $R_i$  and  $R_{i+1}$  that overlap and do not share a common center, with neither contained in the other.*

*Proof.* (1) follows from Lemma 6 and its proof: if  $s_i$  were an element of  $\mathcal{S}_{1 \times 1}$  or  $\mathcal{S}_{m \times n}$  we could obtain a different expression for  $b$  as a product of spins of the same length by “shifting”  $s_i$  either to the left or right. (2) follows from parts (3) and (4) of Proposition 1.  $\square$

We conclude with a list of some open problems:

1. Give a short proof that  $k(3, 3) = 9$ .
2. Determine  $k(4, 4)$ .
3. Determine whether  $\lim_{n \rightarrow \infty} k(n, n)/n^2$  exists, and if so, its value.
4. Analyze the distribution of solution lengths in  $\text{Spin}_{m \times n}$ .
5. Determine which spin types  $\mathcal{S}_{i \times j}$  are equivalent.
6. Determine which combinations of spin types generate  $\text{Spin}_{4 \times 4}$ .
7. Give bounds on the number of boards with unique solutions in  $\text{Spin}_{m \times n}$ .